Numerical Methods for the Solution of Partial Differential Equations: A Review*

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The main purpose of this **note** is to provide a wide view of the different numerical methods for the solution of partial differential equations. We aim that this text can help the reader to be aware of some of actual main trends in this area of knowledge. With in the text, we have included several references to different detailed reviews related to each research sub area of this field.

1. Introduction

Numerical methods for the solution of partial differential equations can be broadly separated into two major groups regarding the Lagrangian and Eulerian descriptions of continuous motion. In **Lagrangian algo-**

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rithms the nodes move with continuum, in **Eulerian algorithms** the nodes stay in place while the the continuum moves through the stationary mesh or through Eulerian coordinate system. Moreover, these numerical methods can be further classified in **mesh-structurated** and **mesh-unstrucrurated** methods. Classical discretization techniques, like finite differences, finite elements and finite volumes, which have been developed within both Eulerian and Lagrangian approaches, emerged as *mesh unstructured* methods.

In recent years **mesh-free** and **mesh-adaptive** methods gained much attention, not only in the engineering but also in the mathematics community. One of the reasons for this development is the fact that mesh-free and mesh-adaptive discretizations are often better suited to cope with geometric changes of the domain of interest, e.g. free surfaces and large deformations, than the classical structured-mesh discretization techniques.

Both mesh structurated and unstructurated techniques have been developed within the frame of the mesh-adaptive approach, thus extending the classical finite element, finite difference and finite volume techniques.

Numerical methods that are used to model partial differential equations with steep solution regions often involves high computational cost if a uniform mesh is used. The family of mesh-adaptive methods -also known as *moving mesh methods*- adapts the mesh to the features of the computed

solution. The nodal density is high in regions of large variation and low in regions where the solution variation is small.

On the other hand, mesh-free algorithms are truly knot base methods. Mesh generation is still the most time consuming part of any mesh based numerical simulation. Typically, more than 70 percent of the overall computing time is spent by mesh generators. Since mesh-free discretization techniques are based only on a set of independent points these costs of mesh generation are eliminated.

2. Mesh Adaptive Methods

The diversity of ways that grids are generated has prevented effectively the development of universal adaptive techniques. Since there are many grid generation techniques, there are also many adaptive grid techniques. A reasonable set of goals for the application of adaptation can be proposed based on those stated in [24], by Y. Kallinderis in the preface to the recent special issue [15], and in remarks by the editors of [13]:

1. The fundamental goal of mesh adaptation must be to reduce spatial discretization error and solution grid dependence.
2. Adaptation should result in quantifiable solution accuracy improvement.
3. Temporal accuracy and conservation should be preserved if required.

4. Additional error introduced by the adaptive algorithm should not reduce benefits significantly.
5. Adaptation should be both automatic and efficient.

These goals/criteria provide initial measures against which adaptation algorithms may be compared.

2.1. Structured and Unstructured Grid Methods

Structured grid methods take their name from the fact that the grid is laid out in a regular repeating pattern called a block. Mesh generated by a structured grid generator is typically all quad or hexahedral. Although the element topology is fixed, the grid can be shaped to be body fitted through stretching and twisting of the block. Algorithms employed generally involve complex iterative smoothing techniques that attempt to align elements with boundaries or physical domains. Really good structured grid generators utilize sophisticated elliptic equations to automatically optimize the shape of the mesh for orthogonality and uniformity. Where non-trivial boundaries are required, "block-structured" techniques can be employed which allow the user to break the domain up into topological blocks. Strictly speaking, a structured mesh can be recognized by all interior nodes of the mesh having an equal number of adjacent elements.

Unstructured grid methods utilize an arbitrary collection of elements to fill the domain. Because the arrangement of elements have no discernible

pattern, the mesh is called unstructured. These types of grids typically utilize triangles in 2D and tetrahedral in 3D. As with structured grids, the elements can be stretched and twisted to fit the domain. These methods have the ability to be automated to a large degree. The automatic meshing algorithm typically involves meshing the boundary and then either adding elements touching the boundary (advancing front) or adding points in the interior and reconnecting the elements (Delaunay). Unstructured mesh generation relaxes the node valence requirement, allowing any number of elements to meet at a single node.

While there is certainly some overlap between structured and unstructured mesh generation technologies, the main feature which distinguish the two fields are the unique iterative smoothing algorithms employed by structured grid generators.

2.2. Mesh Adaptive Strategies

Recent discussions of adaptation have used five categories [13]. Specifically, the different techniques to adapt the Finite Element space (for stationary problems), are

- h-refinement: we enrich the finite element space by (locally) refining the underlying spatial partition The insertion/deletion of mesh nodes resulting in an overall increase/decrease of the number of cells,

- p-refinement is performed by increasing for fixed mesh the polynomial degree of the ansatz space,
- h-p-refinement is a combination of the two last items. Adaptive finite element methods that are capable of exploiting both local polynomial -degree-variation (p-refinement) and local mesh subdivision (h-refinement) offer greater flexibility and improved efficiency than mesh refinement methods which only incorporate h-refinement or p-refinement in isolation,
- r-refinement: one relocates the mesh points in order to get a better resolution of the solution with fixed amount of unknowns. The number of nodes remain constant but are relocated physically in the domain while maintaining identity and data structure,
- m-refinement: one switches to a different equation (= physical model) depending on the local behavior of the approximated solution. As an example one may use linearized equations only if the nonlinear terms of the physical model are negligible.

2.3. Eulerian and Lagrangian Adaptive Methods

In the Eulerian approach, an adaptive mesh technique (h-version, hp-version) must follow the time-dependent features of the data or solution by local refinement and coarsening of the mesh ([2], [4], [14], [20],

[32]). But time-dependent adaptive mesh refinement and coarsening is not simple, especially for three-dimensional (3D) problems. It is quite involved, programming is complicated, data structures are not easy to handle, and the storage overhead is significant. Besides, good local and global error estimators are necessary. Therefore, there exist only a few *unstructured adaptive* programs which are able to handle 3D application-oriented problems with time-dependent change of the geometry, the data, or the solution.

The Lagrangian viewpoint allows the mesh itself to be moved (r-method) ([1], [25], [26]). But an implementation is still cumbersome, since the mesh may become tangled and twisted, elements may collapse, or angles of some elements may degenerate over time due to the movement of the nodes. The proper treatment of these problems is not an easy task, especially in 3D applications.

3. Mesh-free Methods

Generally speaking, there are two different types of **mesh-free approaches** the classical **particle methods** ([29], [30], [28], [31]), and **griddles discretizations** based on data fitting techniques ([3], [7]).

3.1. Particle Methods

Traditional particle methods stem from physics applications like Boltzmann equations [12]. They are truly Lagrangian methods, i.e., they are based upon a time-dependent formulation or conservation law. In a particle method we use a discrete set of points to discretize the domain of interest and the solution at a certain time. The PDE is transformed into equations of motion for the discrete set of particles such that the particles can be moved via these equations. After time discretization of the equations of motion we obtain a certain particle distribution for every time step. Therefore, we get an approximate solution to the PDE via the definition of a density function for these particle distributions. These methods are easy to implement. However, they exhibit in general relatively poor convergence properties in weak norms.

3.2. Gridless Methods

The so-called gridless methods follow a different approach. Here, patches or volumina are attached to each point whose union forms an open covering of the domain. Then, local shape functions are constructed with the help of methods from data fitting. These shape functions are used in a Galerkin or collocation discretization process to set up a linear system of equations. Finally this system must be solved efficiently. In contrast to particle methods, such gridless discretizations may also be applied to

stationary and elliptic problems. According to the data fitting method involved *we can distinguish basically the following three approaches:* **Shepard's method** [33], which has a consistency of first order only, the **moving least squares method** (MLSM) ([18], [19]), which generalizes Shepard's approach implicitly to the case of higher order shape functions, and the **partition of unity p-version method**, which generalizes Shepard s approach explicitly to higher consistency orders. Meanwhile, *different realizations* of these approaches exist. First, there is the **smoothed particle hydrodynamics** (SPH) technique of Lucy and Monaghan ([10], [11], [23], [27], [28]), which resembles (up to area weighted scaling) Shepard s method. Then, Duarte and Oden ([7], [6]) used in their **hp-cloud approach** the moving least squares (MLS) idea. Belytschko and coworkers ([26], [29]) apply similar techniques based on the MLS approach to engineering problems. Furthermore, Dilts, [5] used the MLS technique to extend the SPH method to the so-called **MLS particle hydrodynamics** (MLSPH) method. Babuska and Melenk [28] proposed the so-called **partition of unity method** (PUM), which mainly has been applied to uniform point distributions up to now. Liu, Jun, and Zhang [22] proposed variants of the SPH method based on the idea of **reproducing kernels of higher order and wavelets**. There also exist generalizations of the **finite difference approach to the gridless setting** [21]. Furthermore, Kansa ([16], [17]), Franke and Schaback ([8], [9]), and Wendland [35] used the **radial basis approach** from approximation theory to construct meshless

methods for the discretization of PDEs. The **mass-packet method** of Yserantant ([36], [37]) is somewhat different from the classical particle methods. Here, the particles are not considered in the sense of statistical mechanics but they are understood as comparatively big mass-packets, and the conservation of mass is automatically guaranteed by this ansatz. For an overview on meshless methods see ([34]) and the references therein.

All these data fitting approaches do not depend (at least to a great extent) upon a mesh or any fixed relation between gridpoints (particles). However, the realization and implementation of such a method is not so simple in general: there are often problems with stability and consistency. Furthermore, in a Galerkin method, the discretization of the differential operator, i.e., the integration of the stiffness matrix entries, is in general quite involved in comparison with the conventional grid-based approach. Another challenging task is the discrete formulation of Dirichlet boundary conditions, since the constructed shape functions are in general noninterpolatory. Nevertheless, the different variants of gridless methods are interesting from both the practical and the theoretical point of view. *These methods, which are up to now merely in an experimental premature state, possess some potential and might have an interesting future.*

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